Static spherically symmetric scalar field spacetimes with C^0 matching

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All the classes of static massless scalar field models available currently in the Einstein theory of gravity necessarily contain a strong curvature naked singularity. We obtain here a family of solutions for static massless scalar fields coupled to gravity, which does not have any strong curvature singularity. This class of models contain a thin shell of singular matter, which has a physical interpretation. The central curvature singularity is, however, avoided which is common to all static massless scalar field spacetimes models known so far. Our result thus points out that the full class of solutions in this case may contain non-singular models, which is an intriguing possibility.

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Spherically symmetric solutions of Einstein equations for static massless scalar field configurations have been investigated in considerable detail in past. Bergmann and Leipnik [1] were among the first to construct spherically symmetric static solutions for a massless scalar field. They had, however, only a limited success due to an inappropriate choice of coordinates. Around the same time Buchdal [2] developed techniques to generate solutions for this system, and also Yilmaz [3] and Szekeres [4] found some classes of solutions for the static massless scalar field configurations in general relativity.

Subsequently, Wyman [5] systematically discussed these solutions in comoving coordinates where the energy momentum tensor is in diagonal form, and showed a general method to obtain solutions in the case when the scalar field was allowed to have no time dependence. This gave a unified method to obtain most of the solutions obtained earlier. Also, Xanthopoulos and Zannias [6] gave a class of solutions for time independent scalar fields in arbitrary dimensions where the spacetime metric was static. Further, the static scalar fields conformally coupled to gravity have been a subject of immense interest to many researchers [7], [8], [9], [10]. Static massless scalar fields have also been investigated in settings more general as compared to spherical symmetry (see e.g. [11], [12]).

The main interest in these models has been mainly due to several interesting properties that these solutions exhibit, as was pointed out, for example, by the JNW solution [13]. Mainly these

properties refer to the nature of the spacetime singularity and the event horizons in these spacetimes. There are no trapped surfaces in the model and the singularity which is visible at $r = 2m$ has interesting properties [14].

These features were generalized in [15], leading to a result that for static massless scalar fields, the event horizon is always singular in asymptotically flat spacetimes, and that these results are not necessarily restricted to spherically symmetric models only. In this sense, a study of static massless scalar fields coupled to gravity provides some important insights into the global structure of these spacetimes, and also it gives us useful information on the nature of singularities and trapped surfaces.

The point here is that, the vacuum spherically symmetric model is the Schwarzschild solution, which is a black hole with an event horizon covering the singularity. However, an introduction of a smallest scalar field in the model radically changes the causal properties of the model, making the horizon and trapped surfaces to disappear and the spacetime singularity is visible. It is thus a matter of interest to examine if this class of models admit any singularity free solutions, in order to decide if the presence of a non-vanishing scalar field always causes a naked singularity. While this issue is examined here, in the process we also find a new class of models for static massless scalar field, indicating the possibility of non-singular massless static scalar field models.

One of the main features of the Wyman class of solutions, which is the currently available most general class of models in the case under consideration, is that there exists in these spacetimes a central singularity which is naked. In the present note, we report a class of solutions where there is no such strong curvature singularity. However, a C^0 matching is necessary to achieve this, and as a result there is a shell which has singular matter. Many examples of this type of matching of spacetimes are available in the literature. Usually, this type of singularity can be given a physical interpretation, unlike the strong curvature singularity, and hence it is not considered to be pathological as the former. This type of matching conditions were first introduced by Lanczos and Israel [16]. Later on this has been used by many other authors (for a nice review on the topic, see e.g. [17]). The thin shell formalism has also been used for static spacetimes [18]. In our case also, a C^0 matching is performed in a static case. This class of solutions presented here is different from the Wyman class of solutions and does not have a naked curvature singularity.

We consider here a four-dimensional spacetime manifold which has spherical symmetry. The massless scalar field $\phi(x^a)$ on such a spacetime manifold (M, g_{ab}) is described by the Lagrangian, $\mathcal{L} = -\frac{1}{2}\phi_{;a}\phi_{;b}g^{ab}$. The corresponding Euler-Lagrange equation is then given by, $\phi_{;ab}g^{ab} = 0$, and

the energy-momentum tensor for the scalar field, as calculated from this Lagrangian, is given as

$$T_{ab} = \phi_{;a}\phi_{;b} - \frac{1}{2}g_{ab}\left(\phi_{;c}\phi_{;d}g^{cd}\right). \tag{1}$$

Let us consider the massless scalar field which is a *Type I* matter field [19], *i.e.*, the energy-momentum tensor admits one timelike and three spacelike eigen vectors. At each point $q \in M$, we can express the tensor T^{ab} in terms of an orthonormal basis $(\mathbb{E}_0, \mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3)$, where $\mathbb{E}_0$ is a timelike eigenvector with the eigenvalue ρ and $\mathbb{E}_\alpha$ $(\alpha = 1, 2, 3)$ are three spacelike eigenvectors with eigenvalues p_α. The eigenvalue ρ represents the energy density of the scalar field as measured by an observer whose world line at q has an unit tangent vector $\mathbb{E}_0$, and the eigenvalues p_α represent the principal pressures in three spacelike directions $\mathbb{E}_\alpha$.

We choose the spherically symmetric coordinates (t, r, θ, ϕ) along the eigenvectors $(\mathbb{E}_0, \mathbb{E}_\alpha)$, such that the reference frame is *comoving*, as was chosen by [5] and also [6]. The general spherically symmetric metric can now be written as,

$$ds^2 = e^{2\nu(t,r)}dt^2 - e^{2\psi(t,r)}dr^2 - R^2(t,r)d\Omega^2, \tag{2}$$

where $d\Omega^2$ is the metric on a unit 2-sphere and we have used the two gauge freedoms of two variables, namely, $t' = f(t, r)$ and $r' = g(t, r)$, to make the g_{tr} term in the metric and the T_{tr} component of the energy-momentum tensor of the matter field to vanish. Thus the energy-momentum tensor has a diagonal form. We note that we still have two scaling freedoms of one variable available, namely $t \to f(t)$ and $r \to g(r)$. We note that the variable R represents the physical radius.

We have for spherical symmetry $\phi = \phi(t, r)$, and from equation (1) we see that $T_{10} = \phi'\dot{\phi} = 0$. So we have necessarily $\phi(t, r) = \phi(t)$ or $\phi(t, r) = \phi(r)$, with the energy-momentum tensor being diagonal. For the metric (2), and using the following definitions,

$$G(t, r) = e^{-2\psi}(R')^2, \quad H(t, r) = e^{-2\nu}(\dot{R})^2\ , \tag{3}$$

$$F = R(1 - G + H)\ , \tag{4}$$

we can write the independent Einstein equations for the spherically symmetric massless scalar field (in the units $8\pi G = c = 1$) as below (see [20], [21]),

$$\rho = \frac{F'}{R^2R'}, \tag{5}$$

$$P_r = -\frac{\dot{F}}{R^2\dot{R}}, \tag{6}$$

$$\nu'(\rho + P_r) = 2(P_\theta - P_r)\frac{R'}{R} - P_r', \tag{7}$$

$$-2\dot{R}' + R'\frac{\dot{G}}{G} + \dot{R}\frac{H'}{H} = 0, \tag{8}$$

In the above, the function $F(t,r)$, also called the Misner-Sharp mass, has the interpretation of the mass for the matter field, in that it represents the total mass contained within the sphere of coordinate radius r at any given time t. As noted above, in the static case, the metric components $g_{\mu\nu}$s are functions of r only necessarily, but the scalar field ϕ itself can be in general either r or t dependent. In the case when $\phi = \phi(r)$, which we consider here, the components of the energy-momentum tensor are given by,

$$T_t^t = -T_r^r = T_\theta^\theta = T_\phi^\phi = \frac{1}{2}e^{-2\psi}\phi'^2 \tag{9}$$

It follows that the equation of state in this case, which relates the scalar field energy density and pressures is thus given by $\rho = P_r = -P_\theta$. As noted by Wyman [5], there can be a class of static spacetimes where $\phi = \phi(t)$ also. But we would consider here only the class for which $\phi = \phi(r)$, which describes many earlier known interesting solutions for static scalar field spacetimes.

We shall now consider the static spacetimes, when $\phi = \phi(r)$ and $g_{\mu\nu} = g_{\mu\nu}(r)$. The Einstein equations given above then reduce to the following set of equations,

$$\frac{1}{2}e^{-2\psi}\phi'^2 = \frac{F'}{R^2R'}, \tag{10}$$

$$\frac{1}{2}e^{-2\psi}\phi'^2 = e^{-2\psi}\left(\frac{R'^2}{R^2} + \frac{2R'\nu'}{R}\right) - \frac{1}{R^2}, \tag{11}$$

$$\phi'' = \left(\psi' - \frac{2R'}{R} - \nu'\right)\phi'. \tag{12}$$

$$e^{-2\psi}R'^2 = 1 - \frac{F}{R} \tag{13}$$

In the above, the equation (12) can be integrated once with respect to r to give

$$\phi' = \frac{e^{\psi-\nu+a}}{R^2}, \tag{14}$$

where $a = const$. Eliminating now ϕ' from these equations gives,

$$\frac{1}{2}\frac{e^{-2\nu+2a}}{R^2}R' = F', \tag{15}$$

$$\frac{1}{2}\frac{e^{-2\nu+2a}}{R^4} = e^{-2\psi}(\frac{R'^2}{R^2} + \frac{2R'\nu'}{R}) - \frac{1}{R^2} \tag{16}$$

$$e^{-2\psi}R'^2 = 1 - \frac{F}{R} \tag{17}$$

We note that there is still a freedom left to transform the radial coordinate r, and thus the number of unknown variables is reduced to three in the three equations above. This freedom is just a coordinate transformation of the form

$$r \to \chi(r), \tag{18}$$

which is allowed by the spherical symmetry of the spacetime.

Apart from the above equations, we can also obtain a useful first integral, which is actually contained in the previous equations and it is advantageous to use it. The Einstein equations in this case can be written in the form $R_{\mu\nu} = \phi,_\mu \phi,_\nu$. This implies $R_{00} = 0$, from which we get

$$\nu' = \frac{he^{\psi-\nu}}{R^2} \tag{19}$$

where the quantity h is a constant. From(19) and (14), we get,

$$\nu = \frac{\alpha\phi}{2} + C_1 \tag{20}$$

where $\alpha = he^{-a}$ and C_1 are constants.

To examine the Einstein equations above, we now define a function $f(R)$ as below,

$$f(R),_R = \frac{e^{-2\nu}}{2R^2} \tag{21}$$

The above is a general definition, and not any assumption, because the metric functions here depend on r only. Using this in (15), we get,

$$F = e^{2a}f(R) + C_2 \tag{22}$$

We can choose $C_2 = 0$, which gives $F = e^{2a}f(R)$. Using this in (17), we get

$$e^{-2\psi}R'^2 = 1 - \frac{e^{2a}f(R)}{R} \tag{23}$$

Also, from (21) we get,

$$-2\nu' = (\frac{f,_{RR}}{f,_R} + \frac{2}{R})R' \tag{24}$$

Taking $e^{2a} = 1$ and $C_2 = 0$ for simplicity and clarity of presentation, the last two equations, together with (16) give

$$R(f_{,R})^2 = (f - R)(f_{,R} + Rf_{,RR}) - Rf_{,R} \tag{25}$$

The equation above holds true in generality and we have not made here any assumption of a special coordinate condition or no specific radial coordinate choice has been made. The above represents clearly the main Einstein equation in the static case when $\phi = \phi(r)$. The solutions to the same give the classes of allowed static massless scalar fields models in general relativity. The above is a non-linear ordinary differential equation of second order which is in general difficult to solve fully.

We now consider a particular solution of (25), which is given by $f(R) = -\frac{1}{R}$, which solves the above as is easy to check by inspection. This solution of $f(R)$ gives a class of solutions to the Einstein equations as will be shown now. It should be noted here that for $f(R) = -\frac{1}{R}$, we have from (21), $e^{-2\nu} = 2$ and so $\nu = const$. Then from (19), we get $h = 0$, and therefore $\alpha = 0$ in (20). This shows that ϕ can still be a non-constant function of r even when ν is constant.

From now on, we focus here only the class given by the condition $f(R) = -1/R$. To write down the solution explicitly and to specify the same in terms of the metric components, a choice of the radial coordinate r has to be made now. We note here that in the above consideration, there exists a freedom of choosing $\phi(r)$, and therefore a choice for the same corresponds to actually making a choice of the radial coordinate r. Only after we have made a choice of $\phi(r)$, would that scaling be fixed.

In general, for the case $f = -\frac{1}{R}$, from earlier equations we have,

$$e^{\psi} = \frac{1}{\sqrt{2}} R^2 \phi'(r) \tag{26}$$

Then, the Einstein equation $e^{-2\psi} R'^2 = 1 - \frac{F}{R}$ implies that we have,

$$\frac{2R'^2}{\phi'^2(r)} = R^2(R^2 + 1).$$

By solving the above equation we get,

$$R = \frac{1}{\sinh\left(\pm\frac{1}{\sqrt{2}}\phi(r)\right)} \tag{27}$$

It thus follows that we can now write down explicitly all the metric components, in terms of the function $\phi(r)$ and $\phi'(r)$ above, thus giving a full solution. The metric coefficient ν can be found out from (21). The metric component R^2 can be found out from (27). Finally the function ψ is also known from (26). This completes the solution.

The function $\phi(r)$ here can be viewed as a free function that we are actually allowed choose, and the energy conditions are always respected for all such choices. As an example, putting $\phi(r) = \frac{1}{r}$, we can recover a particular solution from the Wyman class [5], which is given by,

$$ds^2 = \frac{1}{2}dt'^2 - [\frac{r^{-1}}{\sinh{(\frac{1}{\sqrt{2}}r^{-1})}}]^4 dr^2 - [\frac{r^{-1}}{\sinh{(\frac{1}{\sqrt{2}}r^{-1})}}]^2 r^2 d\Omega^2 \tag{28}$$

We note that the range of ϕ is from zero to infinity in this case.

In general, we can in fact recover the entire Wyman class by this procedure. Each solution of (25) with $\phi(r) = \frac{1}{r}$ gives one solution from the Wyman class. The solution set of (25) with this choice of $\phi(r)$ gives the whole Wyman class of solutions. We note here that, in general, the choice of the function $\phi(r)$ is not just a gauge choice. For example, if we take $\phi(r) = -\frac{1}{r}$, the range of ϕ has an upper bound, but no lower bound and it goes from zero to negative infinity and the upper bound can be changed from zero to any other number also. In the choice that Wyman made, the range of ϕ is different, and is necessarily restricted from below in that it goes from zero to positive infinity. Thus, in general the solution would be different from the Wyman class of models.

For the metric given by (28), the Ricci scalar R_c is given by

$$R_c = -\frac{2}{R^4} \tag{29}$$

Therefore there is a curvature singularity at the center $R = 0$. In what follows, we construct a class of solutions where the central singularity is absent, by choosing a specific form of $\phi(r)$. Before proceeding further, however, we need to calculate the quantity $P_l = \int e^{\psi} dr$. This would be necessary to find out the proper length between two shells on any $t = constant$ hypersurface.

$$P_l(r) = \int e^{\psi} dr = \frac{2\sqrt{2}}{1 - e^{\sqrt{2}\phi(r)}} \tag{30}$$

Our purpose here is to construct and find a solution without any strong curvature singularity in the spacetime. Towards that purpose, we need to remove the central singularity $R = 0$, which for example exists in all other models, as discussed above. One way to achieve this is to construct a solution such that the physical radius R in fact does not vanish. To do this, we first notice, from (27), that $R \to 0$ when $\phi(r) \to \pm\infty$. Also from (30), we see that P_l diverges when $\phi \to 0$. Since we are considering a static solution without any strong curvature singularity here, we can consider any $t = constant$ hypersurface where the physical radius R must not vanish. In that case, there is no strong curvature central singularity in the spacetime. Further, since the spacetime is static here, it will be inextendible provided the proper length from any point on the spacelike hyperspace (any $t = const$ surface) to the outer boundary of the hypersurface is infinite. This ensures the

regularity of the solution. If both of these requirements are to be satisfied, then ϕ cannot diverge and must go to zero twice in a range of the radial coordinate r. This implies that $\phi(r)$ must have an extrema somewhere in that range of r where $\phi'(r) = 0$.

There are in fact an infinite number of functions which satisfy these criteria, as required above, and so all of them are equivalent in this regard. Therefore, we make a simple choice here as given by,

$$\phi(r) = (r-a)(b-r) \tag{31}$$

For this choice, ϕ has a maxima at $r = \frac{(a+b)}{2}$ where $\phi'(r) = 0$. From (26), it follows that $e^{\psi} = 0$ at this point, which means that the comoving coordinate system breaks down there. However, in the limit of $r \to \frac{(a+b)}{2}$, the curvature R_c remains finite. Also, the proper length between the shells $r = \frac{(a+b)}{2}$ and $r = b$ is infinite. So it is seen that this coordinate system covers the entire domain,

$$R_{min} \leq R \leq \infty, \tag{32}$$

where $R_{min} = \frac{1}{\sinh[\frac{1}{\sqrt{2}}\frac{(b-a)^2}{4}]}$.

It is clear that the spacetime can be extended through the hypersurface $R = R_{min}$. We do this by joining two identical domains $R_{min} \leq R \leq \infty$ together at the hypersurface $R = R_{min}$. In this case, there is no central curvature singularity in the spacetime and two such identical domains are glued together to give the full spacetime. To examine the matching at this joint, we need to find out the extrinsic curvature at the hypersurface $r = \frac{(a+b)}{2}$. First, we rescale the time coordinate $t' \to \tau$, such that $d\tau^2 = \frac{1}{2}dt'^2$. We consider the orthonormal frame given by

$$\begin{aligned} n_\mu &= (0, |e^{2\psi}|, 0, 0) && (33)\\ e^\mu_{(\tau)} &= (1,0,0,0), && (34)\\ e^\mu_{(\theta)} &= (0,0,1/R,0), && (35)\\ e^\mu_{(\varphi)} &= (0,0,0,1/R\sin\theta). && (36) \end{aligned}$$

The extrinsic curvature of the hypersurface is then given by

$$K_{(A)(B)} = -e^\mu_{(A)}e^\nu_{(B)}\nabla_\mu n_\nu = n_\nu e^\mu_{(A)}\nabla_\mu e^\nu_{(B)}, \tag{37}$$

with $A, B = \tau, \theta, \phi$.

In this case,

$$K_{(\theta)(\theta)} = \frac{|e^{2\psi}|}{e^{2\psi}}\frac{\coth(\frac{1}{\sqrt{2}}(r-a)(b-r))}{R^2}, \tag{38}$$

and

$$K_{(\varphi)(\varphi)} = \frac{|e^{2\psi}|}{e^{2\psi}} \frac{\coth(\frac{1}{\sqrt{2}}(r-a)(b-r))}{R^2}, \tag{39}$$

and the other components vanish. Now we note that,

$$\lim_{r \to \frac{(a+b)}{2} \pm 0} \frac{|e^{2\psi}|}{e^{2\psi}} \coth(\frac{1}{\sqrt{2}}(r-a)(b-r)) = \mp \coth\left[\frac{(b-a)^2}{4\sqrt{2}}\right], \tag{40}$$

and thus the extrinsic curvature of the hypersurface $r = r_{\rm min}$ is multi-valued. This means that there is a distributional source on this hypersurface, which we now calculate. The stress energy tensor of this source is given by $T_{\mu\nu} = S_{\mu\nu}\delta(\eta)$, where η is the Gaussian normal coordinate.

$$S_{(A)(B)} = e^{\mu}_{(A)} e^{\nu}_{(B)} = [(K^{+}_{(A)(B)} - h_{(A)(B)} trK^{+}) - (K^{-}_{(A)(B)} - h_{(A)(B)} trK^{-}) \tag{41}$$

where $\lim_{r \to \frac{(a+b)}{2} \pm 0} K_{(A)(B)} = K^{\pm}_{(A)(B)}$ and $h_{(A)(B)}$ is the induced metric on the hypersurface $R = R_{min}$.

From this it follows that,

$$S_{(\tau)(\tau)} = -2\coth\frac{(b-a)^2}{4\sqrt{2}} = -S_{(\theta)(\theta)} = -S_{(\phi)(\phi)} \tag{42}$$

and the other components are zero.

We note here that the surface energy density of this hypersurface is negative. We can, however, see now that the strong curvature singularity is removed in this case by introducing a C^0 matching at this hypersurface. The solution considered here is topologically different from the Wyman class of solutions, as there is no central shell in this case. There is a thin shell of singular matter in this case with a negative energy density. However, if we consider a sufficiently thick shell which includes the singular thin shell, then the mass inside that thick shell is positive. The negative energy thin shell then reduces the total mass content of the thick shell. In fact, this allows one to give a physical interpretation to the C^0 matching. While there is a mild singularity as discussed above at the joining surface, it can be given a sound physical interpretation (unlike the strong curvature singularity), as already discussed and used in the literature (see e.g. [22]).

We also note here the fact that the spacetime we construct here is asymptotically flat. This can be seen in the following way. We consider only one part of the two identical domains, corresponding to $R_{min} < R \leq \infty$. Restricting ourselves only to this one part of the spacetime, we can consider the coordinate transformation so that the scalar field $\phi(\bar{r}) = \frac{1}{\bar{r}}$. This is the gauge that Wyman used to write down a class of solutions. However, our solution is different from that solution, because the coordinate system used by Wyman covers only a part of the spacetime manifold of the solution

given by us. Also, our solution does not have any strong curvature central singularity, unlike the Wyman model. In the Wyman coordinate system, the metric of our solution takes the form given in (28). From this expression, it is seen that in the limit $\bar{r} \to \infty$, $R \to \infty$, and then the metric becomes Minkowskian. So the part of the spacetime corresponding to $R_{min} < R < \infty$, is asymptotically flat. Similarly, the other part of the spacetime can also be shown to be asymptotically flat.

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